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Asymptotic Laws for the Winding Angles of Planar Brownian Motion

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Using constrained path integrals, we study the winding angle distribution of a two-dimensional Brownian motion around a given point. By a careful analysis of the spectral properties of some Schrödinger-like Hamiltonians, we obtain a generalization of the Messulam–Yor law. Various limiting cases are considered.

KEY WORDS: Brownian motion; winding angle; functional integration.

Much work has been devoted to the problem of the winding properties of random walks or Brownian paths. One of the first motivations was to capture some simple features of polymer entanglements.[1] A related problem arises in the study of magnetic flux line entanglements at the surface of the Sun.[2] More recently, the winding properties of a long polymer chain around an attractive rigid rod were considered in relation with the localization transition exhibited by this system.[3] For a very long chain, it is often convenient to approximate the random walk by a diffusion process. However, as already emphasized in ref. 4, this limit is precisely not at all trivial as far as the winding properties are concerned.

Consider, for instance, the two-dimensional Brownian motion (BM) on the punctured plane P-$\{O\}$. The total angle ϕ wound around O during the time (or length) t is asymptotically distributed as a Cauchy law:[5]

$$\lim_{t\to\infty} P_{\mathrm{BM}}\left(x=\frac{2\phi}{\ln t}, t\right)=\frac{1}{\pi}\frac{1}{1+x^2} \tag{1}$$

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whereas the same winding angle for a random walk (RW) is governed asymptotically by a law whose moments are all finite:[6]

$$\lim_{t \to \infty} P_{\mathrm{RW}}\left(x = \frac{2\phi}{\ln t}, t\right) = \frac{1}{2\cosh(\pi x/2)} \tag{2}$$

The link between these two results can be understood through the work of Messulam and Yor,[7] who have refined Spitzer's law (1) by splitting the total winding angle ϕ into a big winding angle ϕ_+ and a small winding angle ϕ_-. These are defined to be the angles wound around the origin by the Brownian particle when it is respectively outside or inside the unit disk. The joint law of the characteristic function is asymptotically given by[7]

$$\lim_{t \to \infty} E\left(\exp\left[i\lambda_+ \frac{\phi_+}{2\ln(t)} + i\lambda_- \frac{\phi_-}{2\ln(t)}\right]\right) = \frac{1}{\cosh\lambda_+ + (|\lambda_-|/\lambda_+)\sinh\lambda_+} \tag{3}$$

Fourier transformation now shows that the asymptotic law for the big winding angle ϕ_+

$$\lim_{t \to \infty} P^+_{\mathrm{BM}}\left(x_+ = \frac{2\phi_+}{\ln t}, t\right) = \frac{1}{2\cosh(\pi x_+/2)} \tag{4}$$

is the same law as the one that gives the total angle for the random walk (2). It is thus the small winding angle ϕ_- in the neighborhood of the origin that is responsible for the huge difference between Brownian motion and random walk; ϕ_- contributes to the asymptotic law of ϕ only in the continuous limit and here makes all the moments infinite. Of course, if one considers a diffusion process excluding an area around the origin, one recovers the big winding law.[8] It is also interesting to point out that other mecanisms such as self-avoidance can produce finite moments and change the scaling factors.[9,10]

In this communication our purpose is to generalize the Messulam–Yor law by dividing the plane into three concentric zones around the origin. One of our aims is to clarify the role played by the intermediate zone and to get a better understanding of the Brownian motion winding properties. In particular, our approach clarifies the relationship between the occurrence of a scaling variable $(\phi/f(t))$ in the long-time asymptotic regime and the spectral properties at low energy of some associated Schrödinger operators.

To begin, let us first fix our notations and conventions. It is convenient to divide the plane into three concentric zones as follows:

1. $r < R_1$.
2. $R_1 < r < R_2$.
3. $R_2 < r$.

Consider a Brownian particle starting from the point $\mathbf{r}_0$ at time $t = 0$. The probability density $P(\mathbf{r}, t \parallel \mathbf{r}_0, 0)$ to be at the point $\mathbf{r}$ at the time t admits the following functional representation:

$$P(\mathbf{r}, t \parallel \mathbf{r}_0, 0) = \int_{\mathbf{r}(0)=\mathbf{r}_0}^{\mathbf{r}(t)=\mathbf{r}} [D\mathbf{r}(\tau)] \exp\left[-\int_0^t \frac{1}{2}\left(\frac{d\mathbf{r}}{d\tau}\right)^2 d\tau\right] \tag{5}$$

In these units the diffusion constant is $D = 1/2$.

We define $\varphi_\alpha(t)$ as the angle wound around the origin O by the Brownian particle when it is inside the α-zone between the times 0 and t:

$$\varphi_\alpha(t) = \int_0^t \frac{d\varphi}{d\tau} \chi_\alpha(r(\tau))\, d\tau \tag{6}$$

where $r(t)$ and $\varphi(t)$ are the polar coordinates of the Brownian particle, and χ_α is the indicatrix function of the α-zone:

$$\chi_1(r) = \theta(R_1 - r)$$
$$\chi_2(r) = \theta(R_2 - r) - \theta(R_1 - r)$$
$$\chi_3(r) = \theta(r - R_2)$$

Finally, we introduce the joint law of the three winding angles regardless of the final point $\mathbf{r}$, which is the law we are interested in here:

$$P(\phi_1, \phi_2, \phi_3, t \parallel \mathbf{r}_0, 0)$$
$$= \int d^2\mathbf{r} \int_{\mathbf{r}(0)=\mathbf{r}_0}^{\mathbf{r}(t)=\mathbf{r}} [D\mathbf{r}(\tau)] \exp\left[-\int_0^t \frac{1}{2}\left(\frac{d\mathbf{r}}{d\tau}\right)^2\right] \prod_\alpha \delta(\phi_\alpha - \varphi_\alpha(t)) \tag{7}$$

A convenient way to impose the constraints in the path integral is to set

$$\delta(\phi_\alpha - \varphi_\alpha(t)) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\lambda_\alpha \exp\{-i\lambda_\alpha [\phi_\alpha - \varphi_\alpha(t)]\} \tag{8}$$

The characteristic function for the three winding angles at the time t is therefore

$$E\left(\exp\left[i\sum_\alpha \lambda_\alpha \phi_\alpha\right], t \parallel \mathbf{r}_0, 0\right)$$
$$= \int d^2\mathbf{r} \int_{\mathbf{r}(0)=\mathbf{r}_0}^{\mathbf{r}(t)=\mathbf{r}} [D\mathbf{r}(\tau)] \exp\left\{-\int_0^t \left[\frac{1}{2}\left(\frac{d\mathbf{r}}{d\tau}\right)^2 - i\frac{d\varphi}{d\tau}\sum_\alpha \lambda_\alpha \chi_\alpha(r(\tau))\right] d\tau\right\} \tag{9}$$

The action appearing in this path integral is the action of a fictitious particle of unit mass and unit charge moving in the plane and coupled to a vortexlike vector potential:[11]

$$\mathbf{A} = \frac{1}{r}\sum_\alpha \lambda_\alpha \chi_\alpha(\mathbf{r})\, \mathbf{u}_\varphi \tag{10}$$

where $\mathbf{u}_\varphi$ is the unit orthoradial vector. The corresponding Hamiltonian is thus given by

$$H = \frac{1}{2}(-i\nabla - \mathbf{A})^2 = -\frac{1}{2}\left\{\frac{1}{r}\partial_r(r\,\partial_r) + \frac{1}{r^2}\left[\partial_\varphi - i\sum_\alpha \lambda_\alpha \chi_\alpha(r)\right]^2\right\} \tag{11}$$

The characteristic function can thus be expressed in terms of the Green's function of H integrated over the endpoint:

$$E\left(\exp\left[i\sum_\alpha \lambda_\alpha \phi_\alpha\right], t \parallel \mathbf{r}_0, 0\right) = \int d^2\mathbf{r} < \mathbf{r}|\exp(-tH)|\mathbf{r}_0\rangle \tag{12}$$

We expand the Green's function in terms of a complete set of eigenstates of H and integrate over the endpoint $\mathbf{r}$. A partial wave analysis shows that the only states that contribute to the characteristic function have zero angular momentum around O. Since the spectrum is purely continuous, they satisfy

$$\left[-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) + \left(\frac{\sum_\alpha \lambda_\alpha \chi_\alpha(r)}{r}\right)^2\right] \Psi_k(r; \lambda_1, \lambda_2, \lambda_3) = k^2 \Psi_k(r; \lambda_1, \lambda_2, \lambda_3) \tag{13}$$

The characteristic function therefore reads

$$E\left(\exp\left[i\sum_\alpha \lambda_\alpha \phi_\alpha\right], t \parallel \mathbf{r}_0, 0\right)$$
$$= \int_0^\infty r\,dr \int_0^\infty k\,dk\, \Psi_k(r; \lambda_1, \lambda_2, \lambda_3)\, \Psi_k(r_0; \lambda_1, \lambda_2, \lambda_3) \exp\left(-t\frac{k^2}{2}\right) \tag{14}$$

The regular solution of (13) inside each zone can be expressed in terms of Bessel functions:

$$\begin{aligned}
r < R_1: &\quad \Psi_k(r; \lambda_1, \lambda_2, \lambda_3) = \alpha J_{|\lambda_1|}(kr) \\
R_1 < r < R_2: &\quad \Psi_k(r; \lambda_1, \lambda_2, \lambda_3) = \beta[J_{|\lambda_2|}(kr) - (\tan \Delta)\, Y_{|\lambda_2|}(kr)] \\
R_2 < r: &\quad \Psi_k(r; \lambda_1, \lambda_2, \lambda_3) = \cos(\delta)[J_{|\lambda_3|}(kr) - (\tan \delta)\, Y_{|\lambda_3|}(kr)]
\end{aligned}$$

The continuity of Ψ and its derivative at $r = R_1$ and $r = R_2$ allows us to compute α, β, Δ, and δ in terms of k, R_1, R_2, λ_1, λ_2, and λ_3. This gives quite complicated expressions which are not very illuminating.

In practice, we are only interested in the asymptotic limit $t \to \infty$, for which one expects that only the bottom of the spectrum will contribute. The statement that there exists in this limit an asymptotic law, with an *a priori* unknown scaling function $f_\alpha(t)$, is the statement that

$$\lim_{t \to \infty} E\left(\exp\left[i \sum_\alpha \lambda_\alpha \frac{\phi_\alpha}{f_\alpha(t)}\right], t \parallel \mathbf{r}_0, 0\right) \tag{15}$$

is independent of t and $\mathbf{r}_0$. In order to construct the scaling functions, it is convenient to set

$$\begin{aligned}
k &= \frac{y}{\sqrt{t}} \\
r &= u\sqrt{t}
\end{aligned}$$

The charactertistic function then reads

$$\begin{aligned}
E&\left(\exp\left[i \sum_\alpha \lambda_\alpha \frac{\phi_\alpha}{f_\alpha(t)}\right], t \parallel \mathbf{r}_0, 0\right) \\
&= \int_0^\infty u\, du \int_0^\infty y\, dy\, \Psi_{y/\sqrt{t}}\left(uy; \frac{\lambda_1}{f_1(t)}, \frac{\lambda_2}{f_2(t)}, \frac{\lambda_3}{f_3(t)}\right) \\
&\quad \times \Psi_{y/\sqrt{t}}\left(\frac{yr_0}{\sqrt{t}}; \frac{\lambda_1}{f_1(t)}, \frac{\lambda_2}{f_2(t)}, \frac{\lambda_3}{f_3(t)}\right) \exp\left(-\frac{y^2}{2}\right)
\end{aligned} \tag{16}$$

The expansion of the Bessel functions of small arguments

$$\begin{aligned}
\lim_{t \to \infty} J_\nu\left(\frac{yr_0}{\sqrt{t}}\right) &\sim \left(\frac{yr_0}{2\sqrt{t}}\right)^\nu \sim \exp\, -\nu \ln\left(\frac{yr_0}{2\sqrt{t}}\right) \\
\lim_{t \to \infty} Y_\nu\left(\frac{yr_0}{\sqrt{t}}\right) &\sim \frac{1}{\pi\nu}\left[\left(\frac{yr_0}{2\sqrt{t}}\right)^\nu - \left(\frac{yr_0}{2\sqrt{t}}\right)^{-\nu}\right]
\end{aligned}$$

together with the expansion of the coefficients α, β, Δ, and δ finally gives

$$\lim_{t\to\infty} E\left(\exp\left[i\sum_\alpha \lambda_\alpha \frac{\phi_\alpha}{f_\alpha(t)}\right], t \parallel \mathbf{r}_0, 0\right) = \frac{1}{\cosh\lambda_3 + [(|\lambda_1| + \lambda_2^2)/\lambda_3]\sinh\lambda_3} \tag{17}$$

with the scaling functions

$$f_1(t) = f_3(t) = \frac{1}{2}\ln t$$

$$f_2(t) = \left(\frac{1}{2}\ln t\right)^{1/2}\left(\ln\frac{R_2}{R_1}\right)^{1/2}$$

This is the central result of this note.

We now consider this formula in various limiting cases.

1. $\lambda_1 = \lambda_2 = 0$ gives the distribution of the big winding angle:

$$\lim_{t\to\infty} E\left(\exp\left[i\lambda_3 \frac{2\phi_3}{\ln t}\right], t \parallel \mathbf{r}_0, 0\right) = \frac{1}{\cosh\lambda_3} \tag{18}$$

Therefore

$$\lim_{t\to\infty} P_{\mathrm{BM}}^{(3)}\left(x_3 = \frac{2\phi_3}{\ln t}, t\right) = \frac{1}{2\cosh[(\pi/2)x_3]} \tag{19}$$

which is the same distribution as the one found by Belisle[6] on a lattice.

2. $\lambda_2 = \lambda_3 = 0$ gives the distribution of the small winding angle:

$$\lim_{t\to\infty} E\left(\exp\left[i\lambda_1 \frac{2\phi_1}{\ln t}\right], t \parallel \mathbf{r}_0, 0\right) = \frac{1}{1 + |\lambda_1|} \tag{20}$$

All the even moments are infinite.

3. $\lambda_1 = \lambda_3 = 0$ gives the distribution of the middle winding angle:

$$\lim_{t\to\infty} E\left(\exp\left[i\lambda_2 \frac{\phi_2}{f_2(t)}\right], t \parallel \mathbf{r}_0, 0\right) = \frac{1}{1 + \lambda_2^2} \tag{21}$$

Therefore

$$\lim_{t\to\infty} P_{\mathrm{BM}}^{(2)}\left(x_2 = \frac{\phi_2}{f_2(t)}, t\right) = \frac{1}{2}\exp(-|x_2|) \tag{22}$$

An interesting feature of these results is that the asymptotic laws for the big and small winding angles do not depend on the values of R_1 and

R_2. What really matters are the excursions of the paths at infinity or in the neighborhood of the origin. The expression of the scaling function $f_2(t)$ shows that the excursions in the intermediate region give a subleading contribution. This is also apparent when one computes the distribution of the total angle $\phi = \phi_1 + \phi_2 + \phi_3$:

$$\lim_{t \to \infty} E\left(\exp\left[i\lambda \frac{2\phi}{\ln t}\right], t \parallel \mathbf{r}_0\right) = \frac{1}{\cosh \lambda + \sinh |\lambda|} = \exp(-|\lambda|) \tag{23}$$

To conclude, let us show that (22) can be obtained in a different way for a thin intermediate zone:

$$R_1 = R - \frac{e}{2}, \qquad R_2 = R + \frac{e}{2}; \qquad e \ll R \tag{24}$$

Assume that the Brownian particle spends a time T inside this region. The winding angle ϕ_2 is distributed according to the conditional law:

$$P_T(\phi_2) = \frac{R}{(2\pi T)^{1/2}} \exp\left(-\frac{R^2\phi_2^2}{2T}\right) \tag{25}$$

Since the occupation time T is a random variable satisfying the Kallianpur–Robbins law (see, for instance, Pitman and Yor[7])

$$P_K(T, t) = \frac{1}{eR \ln t} \exp\left(-\frac{T}{eR \ln t}\right) \tag{26}$$

Eqs. (25) and (26) lead to the following ϕ_2 distribution:

$$P(\phi_2, t) = \int_0^\infty dT\, P_K(T, t)\, P_T(\phi_2) = \left(\frac{R}{2e \ln t}\right)^{1/2} \exp\left[-|\phi_2| \left(\frac{2R}{e \ln t}\right)^{1/2}\right] \tag{27}$$

i.e., precisely (22) in the limit (24).

As a final conclusion, it is interesting to point out that this method can easily be generalized to more general two-dimensional systems. In particular, the influence of a drift on the scaling properties can be studied along similar lines.[12]

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